2019

Optimizing Timber Harvest Value: A Dynamic Programming Investigation

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Optimizing Timber Harvest Value

A Dynamic Programming Investigation

Independent Study Thesis

Presented in Partial Fulfillment of the Requirements for
the Degree Bachelor of Arts in the
Department of Mathematics and Computer Science at The
College of Wooster

by
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The College of Wooster
2019

Advised by:

Dr. John Ramsay (Mathematics)

THE COLLEGE OF

WOOSTER

Abstract

Operations research models have been and continue to be integral to the implementation of forest management strategies. Fundamental operations research models such as linear and dynamic programming have respective advantages and disadvantages when applied to any forest system. An overview of five fundamental operations research models are provided in terms of a timber harvesting system. Timber production management is a branch of forest management that seeks to optimize the economic benefits of harvesting timber. The time-stage nature of timber harvesting systems makes dynamic programming a particularly useful model for timber production management. Thus, dynamic programming is the primary model discussed in this thesis. Through several constructed examples, the structure and solving process of a dynamic programming model is demonstrated and analyzed. Both discrete and continuous dynamic programming models are discussed as well as varying forest management considerations.

Acknowledgments

I would like to acknowledge Dr. John Ramsay for his guidance throughout the past academic year. There have been challenges in creating this study that have made me very grateful for our frequent meetings. It has been a great experience to discuss mathematics in an atmosphere unlike that of a lecture.

I would also like to recognize the scientists, operation managers, forest service employees, and all others that very willingly took time out of their day to help further my research and point me in a better direction: Dr. Jim Hartman, Dr. Greg Wiles, Jim Menlove, Ryan Yoder, Jeffery Turner, David Watkins, Tracy Roof, Karen DeCristofer, Dr. Aaron Ellison, Dr. Robert Naczi, and Dr. Jim Bissell. Thank you!

Finally, I would like to acknowledge my friends for all of the support they have given me throughout this project, and my family for the opportunity to challenge myself academically.

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CHAPTER 1

Introduction

1.1 Forest Management

The forest is a dynamic system, requiring those who take care of it to understand its interactions with other systems as well as the mechanics within the forest itself. An advanced understanding of the inner and outer workings of the forest ecosystem is crucial to knowing what is best for a specific environment. Thus, the management of a forest takes the time and effort of many people, including those in government, science, and administration. To manage a forest is to be concerned with all of these aspects, and create a management plan which reflects the needs of all parties involved. Depending upon the main purpose of a management plan, forest management teams may differ. For example, those managing the forests of a national park may not have the same exact skills or professional goals as those needed to keep a timber production facility functioning. With the main purpose of timber production being to profit off of wood products, there is a large need for economic expertise at a logging company. National parks also require economic expertise, but efforts are more focused on the upkeep of land and sustaining of jobs. Regardless of purpose, the application of operations research models has been and continues to be of aid when discussing forest management strategies for the present and future.

1.1.1 Modeling Within Forest Management

With environmental concerns skyrocketing over the past few decades, there are now mandatory health standards regarding a managed forest ecosystem and surrounding lands. To address new forest management laws and regulations as well as societal views, forest management modeling now typically builds additional assumptions into previous models. Environmental concerns are not weighted equally in all models, with some models addressing specific environmental concerns to follow federally-enforced ordinances. On the other hand, there are forest management strategies that prioritize an increase in natural development and growth, usually for forest lands that have been protected for this purpose. Forest managers with this task look to decrease the impact of human activity so as not to negatively affect the natural environment. The idea that in order to decrease the impact of human activity, humans need to manage and alter the forest system even more is a bit ironic, but forest management is now needed in many areas for any progress.

One model type which can be used to project future timber production values, but is most often used by forest ecologists to see changes in a forest for overall forest health reasons is simulation models. Trajectories of species organization and other variables are based off of biological information. There is little to no optimization involved in these models, as forest systems are most harmonious with a balance of many interactions, rather than the dominance of one variable such as profit. Ellison [2018] The environmental forest manager does not approach the forest as a source of income, so when using simulation modeling there is no need to find a way in which the most income can be made, and there is little to no importance placed on the monetary value of the forest.

Simulation models are sometimes built upon the assumption that a forest is never disturbed and is always a uniform density. Twery [2004] This does not allow for the adjustment to any unexpected behavior in the forest ecosystem. It is easier to account for uncertainty factors by applying other models, such as dynamic programming or Markov decision processes. The existence of spatial variables in the forest system also presents an issue. These variables, such as size, shape, and relative arrangement, are used alongside non-spatial variables to describe the spatial structure of a forest. There is difficulty in accurately representing the spatial structure of a forest given the innate complexity of a forest system. Kurttila [2001] In recent years, geographic information systems (GIS) and other tools have allowed forest managers to monitor spatial variables while also paying attention to the biodiversity levels of a forest throughout time. Ezquerro et al. [2016] Altogether, there are ways to successfully model a forest in which the priorities of the forest manager are solely environmentally driven.

Adaptive management is another approach to managing a forest system. Adaptive management involves proper ecological treatment of the forest and constant "learning through management where knowledge is incomplete." Allen and Garmestani [2018] The main goals of adaptive management are to reduce any factors of uncertainty in the system and expand the comprehensive knowledge of relations to the system. Allen and Garmestani [2018] Successful adaptive management includes the ability to investigate small alterations of a model, allowing the manager to see a plethora of management options with their respective drawbacks. This sensitivity analysis is not only important to adaptive management, but to other real-world models as well.

Although forest management strategies that prioritize the harvesting of timber as a natural resource seem less environmentally-aware, they can still involve assumptions to produce more environmentally friendly solutions. Concerns within the forest management and larger community do not have to be ignored. There are ways to prevent conditions that indicate poor biodiversity levels in a forest, some of these conditions being low forest connectivity, low genetic variation in plant and animal populations, open conditions throughout a forest, little forest cover, and ill constructed roads throughout a forest. Lindenmayer and Franklin [2002] In fact, new laws and regulations, paired with the development of operations research models and techniques, have produced and increased the integration of biodiversity factors into forest management considerations. Recent operations research techniques and approaches for optimizing economic and environmental benefits within forest management systems are working towards obtaining desired future conditions and increased biodiversity levels. Martell et al. [1998]

In this thesis, the forest management type that will be explored and investigated is timber harvesting, and thus there will be a focus on optimizing economic benefits. Some environmental concerns will be considered and addressed, as the protection of the environment is an objective of most forest managers, but these factors will not be given the most attention. The center of this investigation in modeling timber harvesting systems is to optimize forest value over a time horizon. A background in operations research modeling may be useful. For the understanding of terms and context, a short background on timber harvesting systems is provided.

1.1.2 Timber Harvesting

The original objective of forest management was to maximize the profit of timber production from a certain stand or stands. A stand is a patch of forest that is distinct in composition or structure from surrounding areas of forest. Lindenmayer and Franklin [2002] Before the invention of computers with high computational abilities, growth and yield tables were created to predict the development of a stand's volume in order to know when and how much to cut for optimal profit. These tables can still prove useful when solely concerned with stand volume. Twery [2004] However, the priorities of timber production management are changing, with leading objectives being where and how to cut, compared to when and how much. Ezquerro et al. [2016]

As timber is a natural resource, the profit made from cutting and selling timber is an obvious goal of many forest managers in the industry. Large properties across the world are dedicated to the growing and harvesting of timber. This type of forest management is important for many economies and communities and has been shifting towards using more sustainable practices in order to view the forest as a multiple-use system. Hof [1993] For sustainability reasons, the modeling of timber production now includes variables that the average person might not consider when first thinking about a stand's evolution. Just a few of these variables are road paving costs and timetables, fuel costs, the volume of timber remaining after harvest, biodiversity indices, and costs of harvesting equipment upkeep and replacement. Ezquerro et al. [2016]

With the many variables and complexities of the system, there are difficulties in the process of harvest and planting. While trees can grow across almost any landscape, there are locations that are much more convenient and accessible than others. The conditions of a stand such as wind speeds or the slope of a hill or mountainside need to be taken into account, as these factors may have a large impact on the cost of harvest. The removal of trees by helicopter might be required for stands too isolated or dangerous - the danger possibly applying to the environment as well as humans. HowStuffWorks [2008] Digging up any protective layers of soil, or compacting the soil until it can no longer absorb rainfall runs the risk of negatively effecting the water quality in surrounding areas. Both of these situations must be considered throughout the harvesting process. of Forestry [2009]

The harvesting process begins in the planning stage. Forest managers may collaborate with harvesting contractors to identify a stand to clear-cut or thin, and the best management practice (bmp) for the site. The best management practice changes depending upon the conditions of the stand. Harvesting a stand can take the use a lot of equipment and machinery to cut and transport the timber safely. Examples of possible equipment used are chainsaws, a feller buncher which stabilizes a tree while also cutting it down, and a skidder which transports the cut logs away from the cutting site. There are some highly mechanized systems for timber harvest that arguably consume less fuel per unit of overall output compared to low mechanized systems. Machine and fuel costs are two primary costs that impact the net return from harvest, but the two largest costs of timber harvesting in the southern United States, reported in 2011, are hauling and labor costs. Cubbage and Granskog [1982] Baker et al. [2013] Harvested logs are typically transported to a timber mill facility after being loaded onto large trucks, adding to this hauling and labor cost component.

Certain precautionary measures are taken throughout the harvesting process, given the risk of human and environmental harm. For example, the prediction of where a tree will fall and how hard it will hit the ground is important for the safety of workers as well as animal and plant life. There are also industry standards for the height at which a tree can be cut for easy clean-up, future planting, and site preparation. OklahomaGardening [2016]

After harvest, the land is prepared for planting, utilizing the expertise of a silviculture forester whose goal is to control how the forest grows and develops. Equipment such as a D8 bulldozer breaks the soil loose to a certain depth required and new trees are planted, typically in the dormant season, during colder months. Trees are strategically planted at a planned distance from adjacent trees, and often along the topography of the land instead of in straight rows. OklahomaGardening [2016]

The careful planning and procedure of timber harvests contributes to a more sustainable and environmentally friendly industry. By spending money on practices such as taking precautionary measures when cutting down trees and avoiding the removal of protective soil layers, some assumptions concerning the health of a forest are already built into a model through required harvesting costs. Mandating the completion of certain ecosystem-protecting tasks also mandates the spending of money on these tasks, which then takes away from the overall net return made off of a stand.

Throughout all stages of timber harvesting - harvest itself, preparation for planting, and planting - the forest manager looks to quantify value in the land. This can be done using many different measurements and depends on the state of the trees that are harvested. Standing timber is usually measured in board feet, while harvested logs can be measured in board feet, cubic feet, cords, or units of weight. Descriptions of a couple of these measurements are given as follows:

board foot

a unit for measuring the volume of merchantable timber that a log will yield. 1 board foot is equivalent to a 1 foot by 1 foot by 1 inch piece of wood. The volume of merchantable timber can also be expressed in thousand board feet, abbreviated mbf. Nix [2018]

cord

stacked wood which occupies 128 ft^3 of space. The dimensions for this space are typically 4 feet high, 4 feet wide, and 8 feet long. The actual quantity of wood within a cord is influenced by the diameter and length of the logs, the form of individual logs, the thickness of any bark, and the method of stacking and transporting the cord. The amount of "air space"

> in a cord can be diminished by stacking logs with larger diameters due to the fact that they tend to be straighter and smoother. Compared to using machines, stacking logs by hand also produces more solid wood per cord. Williams [1968]

To estimate the value of merchantable timber in a stand, forest managers across the United States utilize log rules. There are three main log rules - Doyle, International, and Scribner. Each log rule will produce a separate volume estimate for the same stand. Nix [2018] These different estimates are partially due to each rule's individual approach to addressing the taper of a tree. If all trees grew as perfect cylinders, then the calculation of timber volume would be made much easier. But, while the diameter of a tree is smaller towards the top compared to the bottom, a timber volume calculation must be altered from the formula for the volume of a cylinder. A trees taper is influenced by its species and surrounding environmental conditions, which ultimately impacts timber volume estimations. Yoder [2019]

The Doyle rule is the most widely used log rule and is very popular in the southeastern United States. The Doyle formula for calculating the board foot yield of a log is:

$$\text{board feet} = (D-4)^2 * \frac{L}{16}$$

D is the diameter of a log at the small end in inches and L is the length of a log in feet. Cassens The Doyle rule is visibly the most similar log rule to the calculation of the volume of a cylinder: $\pi r^2 h$ where r is the radius and h is the height of a cylinder. Using the variables D and L while also converting to feet, the formula for the volume of a cylinder becomes $D^2 * \frac{\pi L}{576} \approx D^2 * \frac{L}{183}$.

Not as popular, but more accurate than the Doyle rule, is the International rule. A formula for the board foot calculation of a tree using the International rule is as follows, with the same definitions for D and L as when using the Doyle rule.

$$\text{board feet} = 0.04976191LD^2 + 0.006220239L^2D - 0.1854762LD + 0.0002591767L^3 - 0.01159226L^2 + 0.04222222L$$

The International rule board foot calculation is more elaborate. For the sake of time management and efficiency, a handheld device can be employed by forest managers when making volume estimates which computes this calculation. Yoder [2019]

Evidently, the timber harvesting process is very complex. While the actions of planting and harvesting require specialized machines, time for planning, and many hours of manpower, there is a job of determining the value in the harvest and the land itself. To maximize the profit made from harvest, a number of variables can be considered. Taking into account all tree species and climates, successful and efficient models may vary. What produces an optimal net return for one stand is probably not what will yield an optimal net return for a stand located in a different part of the world. Improvements have been made, but there is no single model or approach that has been the best choice for managing all situations and variables.

CHAPTER 2

Model Overviews

2.1 Operations Research Models for Timber Harvesting

Of the numerous types of operations research models applied to forest management, a few of the most common are: linear programming, integer programming, nonlinear programming, dynamic programming, and Markov decision processes. All five of these model types can be utilized for optimizing the net return, or any other variable in timber harvest operations. In general, "an optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints" Winston [2004] Constraints limit the domain of a problem and represent requirements of the model. For optimization models including quite a few constraints, the modeler must be careful that the constraints do not limit the objective function from achieving a solution.

Classic operations research optimization techniques such as linear programming have been employed since the mid-20th century, persisting in relevance and functionality. Ezquerro et al. [2016] The following is a discussion of the applications of linear and dynamic programming, and Markov decision processes, along with possible benefits and drawbacks for each model type. A brief overview of integer and nonlinear programming is included as well.

2.1.1 Linear Programming

A linear programming model (LP) in operations research is an optimization model with a linear objective function. This objective function is subject to linear constraints that limit the possible domain of the model. Linear programming is a popular way to model a system due to the ease of solving for optimal values with a particular algorithm - the simplex method. A variable that is typically optimized in the timber production industry is the net return from harvesting and selling. Large linear programming models allow for the inclusion of many constraints, thus allowing for the inclusion of environmental, industrial, social and economic requirements. The expansion of a linear programming model when optimizing the net return of timber production is important due to the many ways one spends and makes money during the harvesting and selling process.

Variables that indicate high or low biodiversity levels in the forest ecosystem can be optimized as well. However, the net return made from harvesting and selling is usually the variable of most concern to those in charge of the timber harvesting process. Forests are not harvested for the sake of the environment, but for the benefit of the economy and individual or community use. Environmental concerns can still be effectively enforced in a linear programming forest management model through the addition of assumptions and constraints. Forestry laws and regulations and economic requirements can also be expressed in this way.

The issue of clear-cutting large areas of forest at once is an example of an environmental concern for forest managers. The preservation of biodiversity levels is more likely to occur in forests with a smaller total clear-cut area, or in forests that are clear-cut in several smaller patches. The question a forest manager has when addressing clear-cutting issues is "where to cut?" If a large forest is sectioned into stands, the manager wants to avoid harvesting a stand immediately next to, or adjacent to, a stand that was recently clear-cut. By adding adjacency constraints to a linear programming model, a forest manager can impose the rule that a minimum amount of time must pass before a stand can be cut down if it is adjacent to a stand that has recently been clear-cut. Martell et al. [1998] For any stand in the current time period, the value of a variable is set to one if that stand was harvested. For any two adjacent stands, their representative variables must sum to less than or equal to one. Consequently, these variables can either be one or zero. The requirement of solution variables having an integer value makes the problem an integer programming model, an extension of linear programming models. Through adjacency constraints, the existence of a large area of clear-cut forest can be avoided, which reflects well on measurements of biodiversity.

The economic impact of timber mills also needs to be reflected in modeling forest management systems. The constant flow of income from selling timber may be crucial to the development and growth of a city or town's economy, especially in some rural areas. Continuation of economic growth can be established with certainty by including minimum harvest levels. Kaiser and Messer [2012] The volume of trees harvested is set greater than or equal to a minimum required value per time period. This value is adjusted, as the profit a city needs to function is different across different regions. A region may not require minimum harvesting levels if their economy is not dependent upon local timber operations.

The addition of economic and environmental constraints allows forest managers to account for many variables in modeling the net return from timber harvests using linear programming. Nevertheless, the true relationships between variables are not always linear, and the spatial structure of a forest can be more accurately described by nonlinear functions. Thus, a nonlinear programming model may be more suitable in cases where spatial structure is a priority. Kurttila [2001]

Two general assumptions of linear programming models are that areas of land can be grouped in that they have similar-enough characteristics, and that modeling changes of a forest system can be done over a long span of time. Martell et al. [1998] After a solution is found, certain management techniques will be applied to the land groupings, or stands, for each time period. If an area that was assumed to have similar characteristics does not actually have similar characteristics, the management technique applied may not be the best technique for the majority of that area. For example, imagine a stand which consists of a wide range of tree species. Over time, one species of tree becomes ready for harvesting, while all other species take more time to reach peak development for harvest. If that stand was assumed to have similar time-to-harvest periods, a management decision to clear-cut may come at a bad time for a large portion of the trees in that stand. This can be an issue with other models as well. To avoid this error and also to keep a more organized harvesting pattern, timber production plantations may only plant and harvest even-aged stands of the same species.

A linear programming model informs a forest manager what forest management decisions to make across many time periods. Trees grow and live for long periods of time compared to the existence of many living things, and there are many changes within a forest ecosystem just over a few years. Hence, making an assumption about the changes of forest variation, biodiversity, profits, etc. over large time periods may not be a low-impact decision on a model. To confront these assumptions, linear programming models can be adjusted to consider unexpected behavior over long time periods and horizons.

2.1.2 Integer Programming

Mentioned previously, an integer programming model (IP) is an extension of a linear programming model which requires some or all decision variables to have integer values. This particular requirement is necessary for scenarios such as optimizing the number of employees needed in an office at once; a solution of 8.45 employees is not applicable to the real world. Models requiring all variables to be integers are called pure integer programming problems, while only some of the variables are required to be integers in a mixed integer programming problem. Winston [2004]

2.1.3 Nonlinear Programming

Nonlinear programming models (NLP) are applied to systems in which any function included in the objective function or constraints can be nonlinear. Within a forest system, many variables change with time in a nonlinear fashion. Examples are growth rates, the average diameter of a tree, and animal or plant populations. Simplifying the system to a linear programming problem would make the problem easier to solve, but would deliver less accurate results. The difficulty in actually finding a feasible result also increases when working with nonlinear relationships. Winston [2004]

2.1.4 Dynamic Programming

Instead of choosing to make management decisions for each time period that optimize the objective function over one large time horizon as in linear programming, dynamic programming (DP) can be used to find more realistic solutions by making optimal management decisions more frequently. Assessing the forest after a shorter period of time has elapsed provides opportunity to make adjustments to management decisions while the forest is changing. In this way, the larger problem is broken down into several smaller problems. Winston [2004]

With dynamic programming, each "smaller problem" is a stage in a stand's evolution. A stage is often a time period. There is a set of possible states that the stand could be in during a stage, and at each state there is a management decision. These management decisions each correspond to a value which represents the variable that is optimized in the forest system. Each management decision made effects the final optimal value. For example, if the forest system is a private tree farm, the value for each state might be the current monetary worth of the farm at a specific time. At each state value, the forest manager makes a decision that moves the model on to the next stage. For a possible management decision in this next stage, there are a number of possible decisions that could have been chosen previously that allow for this management decision to be made. In other words, there can be multiple paths from one management decision, back to the previous stage.

The many paths that each chain of management decisions creates helps to describe the recursive nature of a dynamic programming model. This recursive nature is what connects all of the management decisions in the optimal set together; the optimal decision at a current state depends on the optimal decision paths from subsequent states. Thus, the decision made at a current state does not depend on the decisions that preceded. Winston [2004] To solve for all optimal states, the last stage's optimal decisions for each state in the stage is found first. The second-to-last optimal decision for each state in the second-to-last stage can then be found because it is dependent upon the last stage's optimal decision paths. The third-to last optimal decisions for each state can then be found, and so on.

By working backwards in this manner, the solution set is found more efficiently. Compared to searching for an optimal solution set out of all possible decision paths, dynamic programming is a technique which may limit the feasible optimal solution set when performing the recursive process. Perhaps the optimal value for all states at a particular stage correspond to the same management decision made in the previous stage. This would indicate that all other management decisions in the previous stage can be ignored. All possible solution sets do not have to be tested in order to find an optimal solution set using a dynamic programming model. Similar to linear programming, the objective function for a dynamic programming model can still be subject to constraints. The constraints may limit how many states are available at each stage, ultimately limiting the number of feasible solutions.

As mentioned, a stage is often a time period. This is especially true in forest management models. However, not all dynamic programming models work to find an optimal solution set across a time horizon. The stagecoach problem is a classic example of a dynamic programming model outside of forest management that does not use time periods as stages. This example can be useful in examining the basic structure of a dynamic programming model and will be presented in the next chapter.

Also in chapters to come are examples of two timber harvesting problems that are well modeled specifically by dynamic programming. These are the rotation and thinning problems. The goal of these problems is to find the optimal stand replacement periods (rotation) or optimal partial harvesting periods (thinning) for a stand or stands. Kennedy [1986] For either the rotation or thinning problems, net return is maximized.

While dynamic programming allows a forest manager to adjust management strategies more frequently, a dynamic programming model does not allow for large uncertainties such as disease and forest fire. Even smaller uncertainties such as whether a stand will have enough time to grow to a certain volume are difficult to calculate using linear or dynamic programming. The long growing periods of trees allow for a higher uncertainty factor, as the longer the stand is not harvested, the longer the stand is at risk of uncertain behavior. Introducing the calculation of probabilities of success or failure can help in modeling uncertainties. Another modeling technique can be of use in order to address these issues - Markov decision processes.

2.1.5 Markov Decision Processes

Often times, forest management models cannot accurately determine the best management strategy for a forest due to uncertainties in the future of the forest ecosystem. All variables within an ecosystem are not easily controlled, and nature does not abide by the requests of forest ecologists, timber mill managers, or anyone, for that matter. Disease and forest fires are natural occurrences within a forest that are healthy for plant and animal life in the long run, but disturb human activities, especially the activities of someone whose job is to harvest and sell timber. The risk of small or large uncertainties pose a threat to the productivity of any type of forest management. Thus, there is an interest in calculating the certainty of a forest to evolve to a specific point. For instance, it would be beneficial to know the probability of a stand growing so that it reaches a larger volume range over one time period, or if that stand is more likely to grow so that it still fits within its current volume range.

A basic property of a Markov decision process is that it uses Markov chains. Markov chain models are defined by a set of states, and the transition probabilities of one state moving to another state over a set period of time. Buongiorno and Gilless [2003] Thus, the structure of a Markov chain is much like a dynamic programming model, where the system can be in one of any number of states at one point in time. For a Markov chain model for forest management, the states could be different volumes of timber within a stand, and the transition probabilities could be the probability of moving from one timber volume range to another over a set time period. These probabilities can be nicely ordered within a matrix, and then used to find volume range probabilities for the same stand in the next time period by constructing an initial state vector and applying matrix multiplication. Buongiorno and Gilless [2003]

If using low, medium, and high volume ranges, a transition probability matrix is structured by matrix T as shown below where the $t_{x,y}$ entry is the probability of starting in volume x and ending in volume y. For $x = 1$ or $y = 1$, the volume is a low range, for $x = 2$ or $y = 2$, the volume is a medium range, and for $x = 3$ or $y = 3$, the volume is a high range. So, the $t_{1,1}$ entry of the matrix is the probability of the stand starting in a low volume range and ending in a low volume range over one time period. Buongiorno and Gilless [2003]

The initial state vector for the problem, p_0, contains the probabilities of the stand being in a low, medium, or high volume at the beginning of the time horizon - at time zero. At this time, the volume of the forest is known, so this initial state vector will contain a probability of one for the correct volume range, and probabilities of zero for all other volume ranges. The state vector, p_i, at any time i is structured as follows, and an example initial state vector is given for a stand that has a low volume range. Buongiorno and Gilless [2003]

$$T = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{bmatrix} \quad p_i = \begin{bmatrix} l & m & h \end{bmatrix} \quad p_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

To find the state vector for the next time period, p_1, the initial state vector is multiplied with the transition matrix T. The resulting probabilities reveal the certainty of the stand being in a low, medium, and high volume range at the next time period. For any time period i, the state vector can be found by taking the previous stage's state vector and multiplying it by the transition matrix. Buongiorno and Gilless [2003]

Transition probabilities, and thus transition matrices, can be altered in order to accommodate certain forest management strategies. If the manager of a forest would like to clear-cut all high-volume stands in the period after they reach high-volume, then the probability of moving from a high-volume stand to a low-volume stand would increase, and the probability of moving from a high-volume stand to a high-volume stand would become zero. For any model considering the possibility of disease or forest fire, the probability of moving up in volume classification will never be one, as there will always be a chance of disease or forest fire moving the stand to a lower volume. While different management strategies can be reflected through separate transition matrices and outcomes for future stand states found, each Markov chain represents one management plan. A Markov decision process is defined as a model where Markov chains are used to represent separate management plans with different outcomes. Buongiorno and Gilless [2003]

CHAPTER 3

Dynamic Programming Investigation

3.1 Modeling with Dynamic Programming

The essentials of dynamic programming were outlined in chapter two. Dynamic programming was described as a problem that has been broken down into smaller subproblems and solved recursively. Each smaller problem is solved one at a time, unlike linear programming where all of the decisions are made at once. Dynamic programming has many advantages compared to linear programming, including particular applicability to the time-stage nature of timber harvesting in forest management and independence from linear relationships. The following discussion outlines the essential parts of dynamic programming through a simple, discrete problem called the stagecoach problem. Then, a timber harvesting dynamic programming model is provided and the application of dynamic programming to forest management is investigated in both discrete and continuous settings.

3.2 The Stagecoach Problem

The stagecoach problem is a popular problem used in order to introduce the fundamentals of dynamic programming. A version of the problem is set up as follows:

A traveler is set to make a journey from Beloit, Wisconsin to Superior, Wisconsin by stagecoach. To get to the end of their travels, the traveler must pass through a number of towns, or intermediate points. Taking the stagecoach from one town to the next costs a certain amount of money, and the traveler would ideally spend the least amount of money overall. What route must the traveler take from Beloit to Superior to minimize the total amount of money spent on travels?

This example stagecoach problem can be visualized by the diagram below:

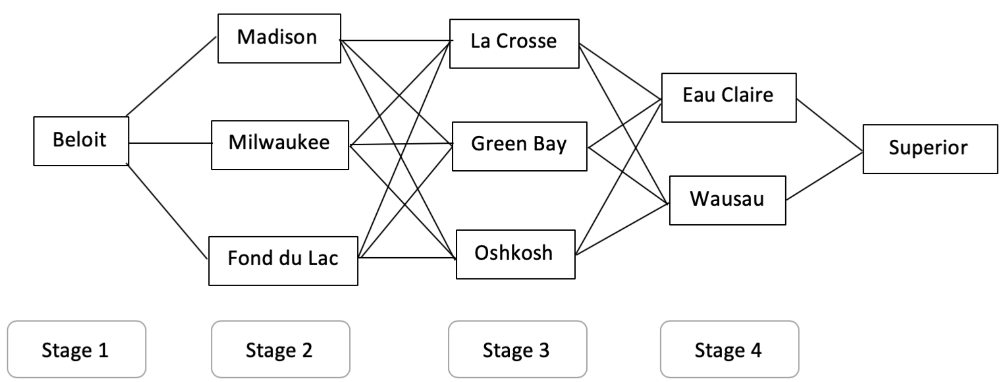

Figure 3.1: Diagram for an example stagecoach problem.

In this stagecoach problem, the traveler is looking to journey from Beloit to Superior and will do so by traveling through three other Wisconsin cities. Evidently, there are no time periods in which the travel needs to occur, so in this example the stages of the problem are not time periods. There are four stages in this example, each with a decision to make regarding the next city to visit. The different cities in each stage are separate states, one of which will be included in the solution set for the optimal path. As seen in Figure 3.1, it is possible to journey from the city the traveler is in to any other city in the subsequent stage.

To minimize the total amount of money spent on travel, each path from city to city needs to be given a monetary value. These values are given by the tables below:

	Madison	Milwaukee	Fond du Lac
Beloit	5	7	11

Table 3.1: Path values for stage one of the stagecoach example.

	La Crosse	Green Bay	Oshkosh
Madison	6	10	5
Milwaukee	9	8	6
Fond du Lac	8	4	2

Table 3.2: Path values for stage two of the stagecoach example.

	Eau Claire	Wausau
La Crosse	6	11
Green Bay	9	5
Oshkosh	11	7

Table 3.3: Path values for stage three of the stagecoach example.

	Superior
Eau Claire	9
Wausau	12

Table 3.4: Path values for stage four of the stagecoach example.

At first it seems like every possible path from Beloit to Superior needs to be compared in order for the path of least cost to be found. By doing so, the optimal path will be found. However, as problems grow, this brute force method demands too much computational time. It is impractical to add up all of the costs for every path when there are a large number of stages and states included in the problem. The brute force method would not take a long time for the stagecoach problem provided, but dynamic programming proves to be more efficient. Ramsay [2017]

The stagecoach problem can also be solved by using a greedy algorithm technique. This technique chooses the current stage's best decision to make for optimality to the next stage, regardless of the effects that the decision might produce further down the path. So, for example, the first decision is to travel from Beloit to Madison since that is the cheapest route to the second stage. From Beloit to Superior, the greedy algorithm would produce the path:

$$\text{Beloit} \rightarrow \text{Madison} \rightarrow \text{Oshkosh} \rightarrow \text{Wausau} \rightarrow \text{Superior}$$

The total cost is 29 units. This path is in fact not the overall optimal path for the traveler to take, but turns out to be close to optimal. Note that the path of most cost to the traveler is the path:

$$\text{Beloit} \rightarrow \text{Fond du Lac} \rightarrow \text{La Crosse} \rightarrow \text{Wausau} \rightarrow \text{Superior}$$

This path yields a cost of 42 units. In this case, the greedy algorithm is not a bad strategy to use. Ramsay [2017]

The greedy algorithm is an example of a heuristic method, which aims at solving a problem quickly rather guaranteeing an optimal solution. Hence, a heuristic method may produce a good, but non-optimal solution, and may even produce a non-optimal and undesirable solution. For the greedy algorithm, the optimal state chosen in one stage may not be a part of the optimal set of states, possibly causing the overall optimal path to be overlooked. This algorithm only considers the immediate state of the problem in making decisions, ignoring any possible long-term consequences. In forest management, the consideration of long-term consequences is crucial for any economic or environmental goals. For this stagecoach problem example, the path found by the greedy algorithm is not the optimal path. Although the greedy algorithm may be quite fast, it is not guaranteed to produce an optimal solution, and is not deemed a good method for finding optimal solutions compared to modeling with dynamic programming. Ramsay [2017]

Now, let us apply the dynamic programming technique explained in chapter two to this stagecoach problem. First, the optimal decisions for each state in the last stage are obtained. For the two states in stage four, there is only one path to end in Superior, so these two paths are optimal for their respective states. Comparing the two paths in this last stage tells us that it costs less to travel from Eau Claire to Superior than from Wausau to Superior. Note that this does not indicate that the path from Eau Claire to Superior will be part of the optimal path when considering all decisions at every stage. Thus, the optimal path when in state Eau Claire is to pay 9 units to go to Superior, and the optimal path when in state Wausau is to pay 12 units to go to Superior. The cost and future state of the optimal decision for each state in stage four is then stored so that they can be easily referred to when using backwards recursion. This data can be found on Figure 3.2 along with the path values, or cost of traveling from one town to the next for stage four.

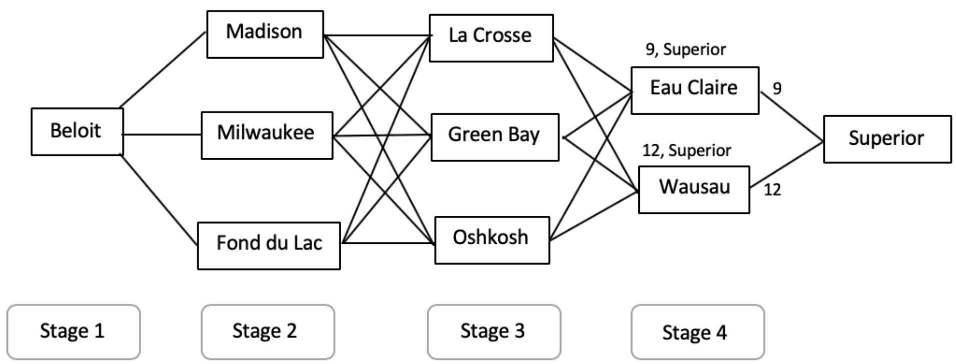

Figure 3.2: Diagram for an example stagecoach problem with path values and stored optimal state values for stage four.

Moving back to stage three, the optimal decision is found again for each state in the stage. The optimal value at each state is equivalent to the minimum costing path out of all paths, or decisions, that could be made to go to the next stage and then finish in Superior. For instance, the optimal value for state La Crosse is the minimum costing path between the two possible paths to get to the fourth stage and then finish in Superior. Going from La Crosse to Eau Claire costs 6 units, plus the cost to finish which is 9. Traveling from La Crosse to Wausau costs 11 units, plus the cost to finish which is 12. Thus, the optimal value for state La Crosse is 15 units by traveling through Eau Claire. (6 + 9 = 15 units < 11 + 12 = 23 units) This data is then stored for the next step as shown above state La Crosse in Figure 3.3. The optimal values for the other two states in stage three are also calculated using the optimal values found for the states in stage four. The results are: 17 units by going through Wausau for state Green Bay and 19 units by going through Wausau for state Oshkosh. This data can again be viewed on the figure below.

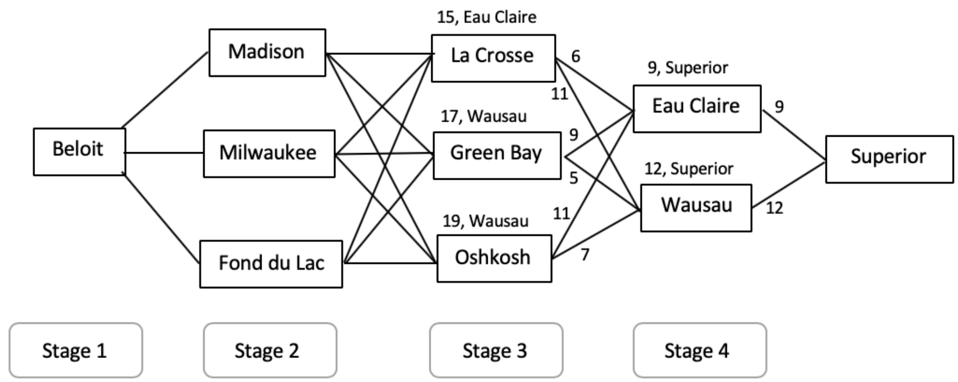

Figure 3.3: Diagram for an example stagecoach problem with path values and stored optimal state values for stages three and four.

Given this stage three information, it is possible to find the optimal values for each state in stage two and for the state in stage one as well. In stage one, the only state, Beloit, turns out to yield an optimal value of 26 by way of Madison. The optimal solution set which minimizes overall cost is found as follows, costing 26 units.

$$\text{Beloit} \rightarrow \text{Madison} \rightarrow \text{La Crosse} \rightarrow \text{Eau Claire} \rightarrow \text{Superior}$$

A final diagram, Figure 3.4, with all stored and path values is provided on the following page.

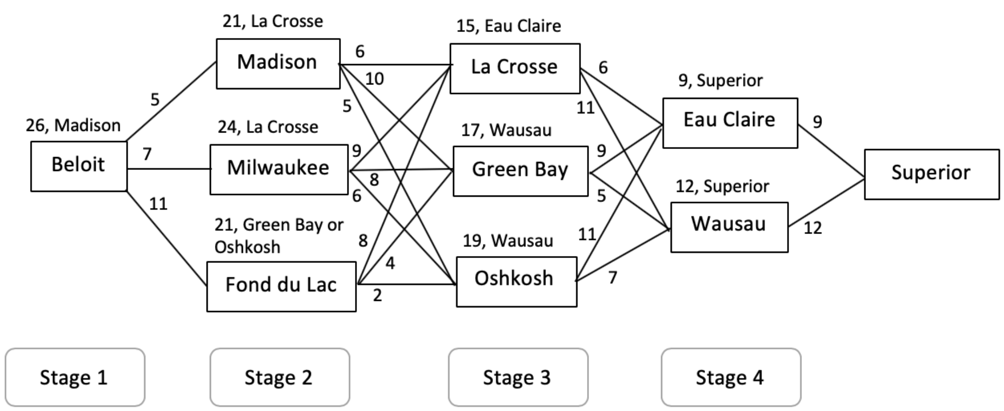

Figure 3.4: Diagram for an example stagecoach problem with all path values and stored optimal state values.

The decision path found during the process of searching for optimal values reveals the recursive behavior and constant assessment of a dynamic programming model. For any state s in stage n, the optimal cost of the system from that state on is the optimal value of moving to the next stage from the current state plus the cost of the optimal path to finish from there. Consequently, the total cost of the system from state s does not depend upon previous decisions or states visited. In other words, the decision made in one stage does not have any sort of impact on the decision made in the next sequential stage. The system is fully assessed during each new stage. In the example provided, due to the backwards recursive process for dynamic programming models, the fact that the traveler went from Beloit to Madison during stage one had no effect on the traveler's decision to go from Madison to La Crosse in stage two. Ramsay [2017]

3.3 Example Discrete Timber Harvesting Model

The stagecoach problem is a discrete model - one that has a finite number of stages and states. Discrete models can grow to be very large problems, with hundreds or thousands of stages and states. Thus, the computation of large models can be difficult and time-consuming. Due to the complexity of forest systems and the influence of many variables, solving for optimal solutions usually involves the use of a computer program that can complete the computations necessary for decision-making quicker than the average human. For simplicity, a small optimal rotation dynamic programming example is given. The problem is set up as follows:

A forest manager uses an area of land with room for 100 trees for timber growth and harvest. The trees in this area are assumed to always be the same age. At 20-year time intervals, the forest manager must decide whether to let the trees grow, or to clear-cut and replant the stand. If the stand is left alone to grow, there is still a present value of the stand, found by considering the stand's future value. Even without selling the available timber, there is still value in the property given that there will be even more timber to sell as time progresses. There is not only value in the timber that can be sold, but value in the potential of the property, which is what the present value accounts for in this case. To calculate present value, the future value of the stand is discounted by 40 percent, assuming that there is an increase in the price of timber sales over time. Essentially, this "discount factor" (0.6) adjusts the future value of the stand to estimate the present value by accounting for inflation.

If the stand is clear-cut, all of the trees are sold and replanting occurs. The net return per tree from clear-cutting, harvesting, transporting, and selling is 1 unit per year the tree has lived. This net return also incorporates the future value of the stand, discounted to account for inflation. Replanting all 100 trees costs 20 units per tree ($2,000$ units). The time horizon begins with 100 newly-planted trees, and if the trees grow to be 80 years old, the stand must be clear-cut. What management decisions must the forest manager make over an 80-year time horizon in order to find the optimal rotation period of the stand? That is, what is the optimal amount of time to leave in between clear-cuts to allow for maximized net return in the stand? Kennedy [1986]

A diagram for this problem is given below:

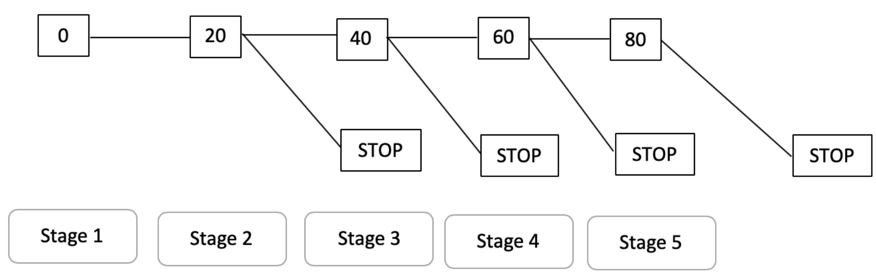

Figure 3.5: Diagram for an example discrete timber harvesting model.

For this example, the stages are 20-year time periods and the states in each stage are all viable ages of the trees in the stand. In stage one, the only management decision possible is to let the trees grow because the stand begins with 100 newly-planted trees at age zero. The possible management decisions in stage two are to clear-cut and replant, returning the trees back to age zero and thus finding the optimal rotation period (indicated by "STOP" in the next stage), or letting the trees grow to age 40. All other stages include the decision to either clear-cut and replant or let the trees grow, excluding the final decision in stage five which must be to clear-cut the trees at age 80. If the stand is ever clear-cut, then the optimal rotation period has been determined, and continuing with the model is redundant because the objective is to find the amount of time in between clear-cuts that allows the manager to make the most return. Continuing with the problem would lead to clear-cutting and replanting after every optimal amount of time has passed, which is the same amount of time.

As with the stagecoach problem, the optimal decisions for each state in the last stage are found first. The last stage of the problem, stage five, occurs at the end of the 80-year time horizon, when the stand must be clear-cut. As this is the only path that can be taken from the 80-year-old stand state, it is the optimal path for the state. Clear-cutting 80-year-old trees yields a return of $8,000$ units, and a replanting cost of $2,000$ units, so the net return is $6,000$ units. An updated diagram with the path value and stored optimal state value for stage five is displayed below.

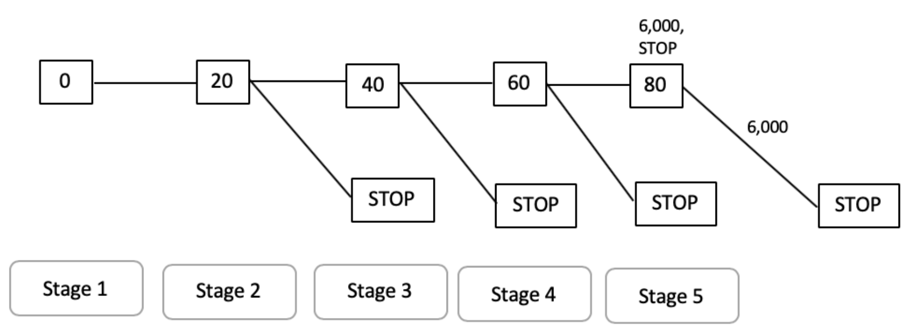

Figure 3.6: Diagram for an example discrete timber harvesting model with the path value and stored optimal state value for stage five.

Moving backwards, the fourth stage of the problem includes making a decision between letting the trees grow to 80 years old, and clear-cutting 60-year-old trees. The money made from clear-cutting the stand in the fourth stage is $6,000$ units, minus $2,000$ for replanting, equals $4,000$ units. If the trees are left alone to grow, then the value of the stand is the discounted future value of the stand. The future value is equivalent to the return made after letting the trees grow to 80 years old, and then clear-cutting, so the discounted future value is $0.6 * 6,000$ units $= 3,600$ units.

Out of the two possible management decisions for the 60-year-old stand state, the optimal path is to return the stand back to age zero. Thus, the optimal management decision for the fourth stage is to clear-cut as seen in Figure 3.7, indicating that the optimal rotation period is at most 60 years.

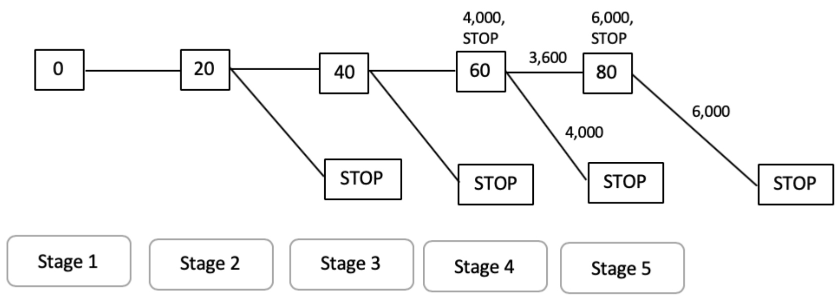

Figure 3.7: Diagram for an example discrete timber harvesting model with path values and stored optimal state values for stages four and five.

To continue with the dynamic programming steps, the model moves backwards to the third stage. This stage of the problem includes making a decision between the same two management decisions in the fourth stage, but with a 40-year-old stand state. Clear-cutting 40-year-old trees yields $2,000$ units $(4,000 - 2,000)$. The value in letting the stand grow is $0.6 * 4,000$ units $= 2,400$ units. Hence, a decision to let the stand grow is optimal for the 40-year-old stand state. The second stage includes a 20-year-old stand state with an optimal decision of letting the stand grow. The state in the first stage has one path available, just like the fifth stage, so the optimal decision for that state is to let the stand grow. A final diagram, Figure 3.8, with all stored and path values is given below.

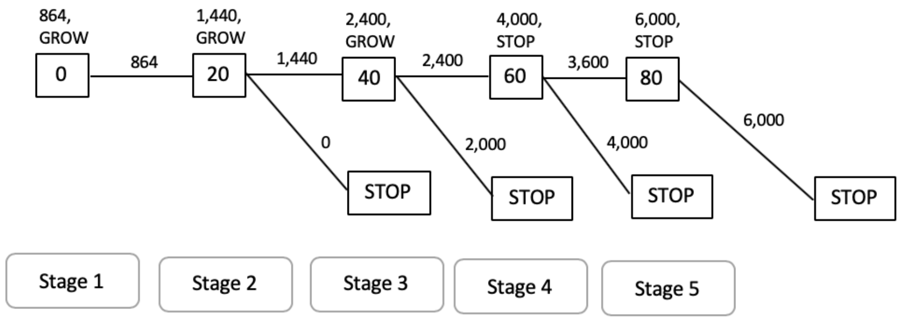

Figure 3.8: Diagram for an example discrete timber harvesting model with all path values and stored optimal state values.

Up until the stand reaches an age of 60 years old, the optimal decision for the stand is to let the trees grow. At 60 years old, it becomes more profitable to clear-cut. After 80 years of growing, the most profitable management decision would also be to clear-cut even without the requirement to clear-cut. If the time horizon was more than 80 years long, then any state in stages following stage five would also include the optimal decision to clear-cut. As the amount of time left for the stand to grow increases, the profit made off of clear-cutting the stand increases. For example, clear-cutting in stage four yields a profit of $4,000$ units and clear-cutting in stage five yields a profit of $6,000$ units. This means that if it is possible to wait longer to clear-cut, the forest manager should wait for a larger return. Although clear-cutting at a stand age of 80 years would be more profitable, the optimal rotation period is 60 years because the landowner makes a profit more frequently.

Altogether, the optimal amount of time to leave in between clear-cuts to allow for maximized profit for this problem is 60 years. In other words, the management decision to clear-cut would come every 60 years for this particular stand of 100 trees. Other constraints or assumptions could be included in this dynamic programming model in order to make the model more realistic. These aspects may be added in order to abide by an economic or social requirement, to describe forests with more than one stand or tree species, to apply more accurate net return functions, etc. While constraints and assumptions help to model the system after the way things work in the world, they also complicate the system. The more constraints and assumptions, the more complex the problem. When these problems become larger and more complex, the ability to compute an optimal solution set can come into question.

A specific alteration that would add to building a more representational model would be to measure the volume of merchantable timber within the stand instead of quantifying the stand by number of trees. Although the trees in the stand are assumed to be the same age, this does not imply that there is an equal amount of timber that can be sold from each. Especially if this stand consists of trees of different species, each tree will not have the same volume. If the stand consisted of trees of different species, the net return from selling each tree would not be the same number of units per year the tree has lived, requiring different net return values for each species.

Next, the difficulty in solving a larger, more complex dynamic programming model is illustrated through an investigation of a continuous timber harvesting model. While assumptions are still made, the model addresses many known difficulties with decision-making for timber harvesting and utilizes real-world values.

3.4 Continuous Timber Harvesting Model

As stated previously, discrete timber harvesting models can grow to be very large, with many states and stages. Even if a discrete problem grows to have hundreds of thousands of states and stages, the number of states and stages is still finite. Discrete problems most accurately describe a system which changes, and then stays in one state for a duration of time. Continuous models, on the other hand, can describe problems with an infinite number of states or stages. By including infinite stages in timber harvest models, which means infinitely many time periods, the management area is assessed and altered constantly. Timber harvest models with infinite states allow for endless management possibilities. While forests are constantly changing in volume, timber quality, and more, a continuous model is more fitting as opposed to a discrete model. In this section, a dynamic programming model with continuous states is considered. Similar to the previous model, the stages of the model will be discrete, mandating the reassessment of a stand after a set time period. All variables will be defined in terms of a fish harvesting scenario presented by Kennedy [1988]. The model will then be applied to timber harvesting.

An article published in 1988 by John O. S. Kennedy includes a discussion of both n-period and infinite-period models regarding natural resource management. Kennedy considers an n-period fish harvesting model in order to deduce optimality conditions for dynamic resource problems. The harvesting level is maximized in order to produce a maximal present value for stage i, $V_i(x_i)$, across the time horizon. The harvesting level is the biomass of fish harvested in each time period. Biomass is a measure of the quantity of desired material, in this case fish, present within a given volume or area. For any time period i where $i = 1, 2, ..., n$, there is a certain biomass of fish stock available, x_i, and a biomass that is actually harvested during period i, represented by u_i. Variable x_i is continuous and will represent the state at any period. Kennedy [1988]

Management decisions for the optimal biomass harvested are set to be made over n time periods. The net return for the fish harvested in time period i is equivalent to the amount of money made from selling the harvested fish, u_i, minus the cost of harvesting those fish. The price per biomass unit that the fish will be sold at in time period i is represented by p_i and the cost per biomass unit is a function of both x_i and u_i, represented by $c_i(x_i, u_i)$. The cost of harvesting fish may include many factors just as in timber harvesting, and is a function of both x_i and u_i because there is a cost per unit of biomass harvested as well as a change in cost depending on the density of the fish, or the biomass of fish per area or volume. A crowded fish farm may have different costs compared to a sparse one. Net return in time period i with respect to the biomass harvested out of the biomass available is profit minus cost and is represented by the function $A_i(x_i, u_i) = p_i u_i - c_i(x_i, u_i)$. Therefore, all variables and functions x_i, u_i, p_i, and $c_i(x_i, u_i)$ contribute to the calculation of net return in each stage. Kennedy [1988]

Over time, the biomass of fish grows through reproduction and individual fish growth and death. This rise in population, and thus rise in biomass over one time period i is described by the function $g_i(x_i)$. From one time period to the next, the biomass of the fish stock increases by the biomass grown in the last period and decreases by the biomass harvested in the last period. This is represented by the relation $x_{i+1} = x_i + g_i(x_i) - u_i$. This formula produces the state variable for the next stage, given the the current stage's state variable, growth quantity, and harvest quantity.

Knowing how to calculate the biomass of the fish stock in the upcoming time period is useful for calculating the future value of the fish stock. The present value of the fish stock when harvested at any level includes future value as well as net return. There is value in any fish stock that is not sold as well as in the potential of the system to produce more fish. This future value is represented by a function for the value of the biomass of the fish stock in the next time period, adjusted by multiplier α for inflation, given by $\alpha * V_{i+1}(x_{i+1})$. The "terminal stock" value is symbolized by the function $F(x_{n+1})$ and is assumed to be equivalent to $V_{n+1}(x_{n+1})$, the value of the fish stock in the $n+1$ time period. Kennedy [1988]

As mentioned, the end goal of the continuous model is to find optimal harvesting levels at all stages to yield optimal present values $V_1(x_1), V_2(x_2), ..., V_n(x_n)$. Considering all variables and functions, we have the dynamic programming model: $V_i(x_i) = \max\limits_{u_i}[A_i(x_i, u_i) + \alpha V_{i+1}(x_{i+1})]$. This model illustrates how the present value of the current decision is determined by the net return of the current decision plus the value to finish from the next stage. In the last stage, n, the present value is $V_n(x_n) = \max\limits_{u_n}[A_n(x_n, u_n) + \alpha V_{n+1}(x_{n+1})]$. $V_{n+1}(x_{n+1})$ is found by the assumption that $V_{n+1}(x_{n+1}) = F(x_{n+1})$, as $F(x_{n+1})$ is known. Optimal u_n is found with a corresponding management decision for each feasible x_n. Consequently, all optimal $V_n(x_n)$ are stored for each state. While the states are continuous, function optimization must be utilized. Every u_n cannot be tested discretely as in the previous dynamic programming examples. Kennedy [1988]

Stepping backwards to the $n-1$ stage, V_n is used in the maximization of u_{n-1} for all feasible x_{n-1}, and all V_{n-1} are stored for each state. This process continues until optimal values are found for all states in stage one and an optimal path is determined. A final set of all V_i and corresponding u_i make up the optimal solution. Kennedy [1988]

This fish harvesting scenario can easily be applied to a timber harvesting system where a certain quantity of timber, u_i, is harvested instead of a certain biomass of fish for any period i. Kennedy [1988] Biomass will be replaced by quantity of timber, measured in thousand board feet, or mbf. A summary of all variables and functions with timber harvesting definitions is given below:

$\mathbf{x_i}$: total quantity of timber in period i

$\mathbf{u_i}$: quantity of timber harvested in period i

$\mathbf{V_i(x_i)}$: optimal present value of timber, dependent upon the quantity of timber

$\mathbf{A_i(x_i, u_i)}$: net return from harvesting u_i from x_i in period i

$\boldsymbol{\alpha}$: discount factor, adjusts future value based off of todays market and future inflation projections

$\mathbf{p_i}$: price of timber in period i

$\mathbf{c_i(x_i, u_i)}$: cost of harvesting u_i from x_i in period i

$\mathbf{g_i(x_i)}$: growth in quantity of timber in period i

$\mathbf{F(x_{n+1})}$: value of the terminal timber quantity, dependent upon the total quantity of timber in the time period following the last stage

The model is structured as follows:

$$V_i(x_i) = \max_{u_i}[A_i(x_i, u_i) + \alpha V_{i+1}(x_{i+1})] \tag{3.1}$$

$$\text{where } A_i(x_i, u_i) = p_i u_i - c_i(x_i, u_i) \tag{3.2}$$

$$x_{i+1} = x_i + g_i(x_i) - u_i \tag{3.3}$$

$$V_{n+1}(x_{n+1}) = F(x_{n+1}) \tag{3.4}$$

The maximization in Equation (3.1) is the sum of the net return from harvest and the optimal present value in the next time period, scaled by the discount factor. The breakdown of net return from harvest is provided by Equation (3.2), and Equation (3.3) is the calculation of the total quantity of timber in the next time period. The final equation, (3.4), exhibits that the present value of timber in the $n + 1$ time period is equivalent to the value of the terminal timber volume.

This allows $F(x_{n+1})$ to be substituted for $V_{n+1}(x_{n+1})$ when finding $V_n(x_n)$. Dynamic programming starts by solving for optimal $V_n(x_n)$ by optimizing u_n. Working backwards, maximal present values across all time periods are found when all optimal levels of harvesting, or quantity of timber harvested, are set for all feasible x_i. The objective of this timber harvesting problem is to find optimal $u_1, u_2, ..., u_n$ to produce optimal $V_1(x_1), V_2(x_2), ..., V_n(x_n)$.

CHAPTER 4

Application of Continuous Timber Harvesting Model

The continuous timber harvesting model introduced in the last chapter contains many moving parts. Several separate functions feed into the optimization of harvesting levels and present value for each time period. For timber harvest stands with more than one species, the functions for timber price, cost, and growth become even more complex. To avoid further complexities, a stand containing only one species will be analyzed, one containing pinus taeda, commonly known as loblolly pine. Carey [1992]

Loblolly pine can be found across most of the southeastern United States, reaching up into southern New Jersey and out into eastern Texas. The species grows rapidly and tends to have a very tall, straight trunk. Mature loblolly trees can grow to exceed 110 feet in height and 30 inches in diameter. These factors make loblolly ideal for timber harvesting. Carey [1992]

In order to build a model which accurately replicates the harvesting of a loblolly stand over time, values for all variables and functions were found using an abundance of sources. What follows is a discussion of these variables and functions, including the price of timber, the cost of harvest, the discount factor, timber volume growth, and the value of the terminal timber volume. For all functions in the model, the total quantity of timber in period i, x_i, is the volume of loblolly present measured in thousand board feet (mbf). The volume of loblolly harvested during that time period in mbf is u_i. Each stage of the problem covers a five-year span, and all states, x_i, from zero to a potential maximum volume of M mbf, are reviewed for optimality in all time periods. The model assumes that the beginning management year is 2018 and the end year is 2058, covering a total of 40 years. Thus, $i = 1, 2, ..., 8$ and $n = 8$. The loblolly stand that is managed is one acre of land.

4.1 Discussion of Applied Variables and Functions

Chapter three provided a summary of all variables and functions used in the continuous model with timber harvesting definitions. In this section, these variables and functions, given below, will be assigned values specific to a loblolly pine stand in the southeastern United States. Every function will be left in terms of u_i and x_i. To begin, the stumpage price, p_i, is determined, which is most plainly the price per volume of timber.

$$p_i \qquad c_i(x_i, u_i) \qquad A_i(x_i, u_i) \qquad \alpha \qquad F(x_{n+1}) = F(x_9) \qquad g_i(x_i)$$

4.1.1 Stumpage Price

The stumpage price at which any kind of timber is sold varies based off of factors such as timber quality and accessibility. Estimations of this price can be made based on south-wide average stumpage prices, which can then be used for p_i in the continuous model. Current price reports from the South Carolina Forestry Commission provide a south-wide average of \$179.7/mbf for pine sawtimber during the last quarter of 2018. Ten years before, during the last quarter of 2008, the south-wide average stumpage price of pine sawtimber was \$218.7/mbf. This decline in average south-wide stumpage prices over the last ten years could be partly due to efforts made to recover from the financial crisis of 2008. Stumpage prices for pine sawtimber have leveled out to around \$179.7/mbf since then. A graph of south-wide pine sawtimber stumpage prices over the last ten years is given below (see the blue line). Anon. [2018] Note that stumpage prices are shown in US$/ton. These numbers have been converted to US$/mbf to arrive at the values above. The conversion calculation was made using the conversion recommended on the South Carolina Forestry Commission's website ("multiply reported price by 7.5"). Anon. [2018]

$\boxed{\text{Set } p_i = 179.7}$

Southeastern Average Stumpage Prices - US$/ton

	one year ago			ten years ago	
	4Q18	4Q17	up/(dn)	4Q08	up/(dn)
Pine Sawtimber	$23.96	$23.59	$0.37	$29.16	($5.20)
Mixed Hardwood Sawtimber	$31.93	$31.45	$0.48	$23.12	$8.81

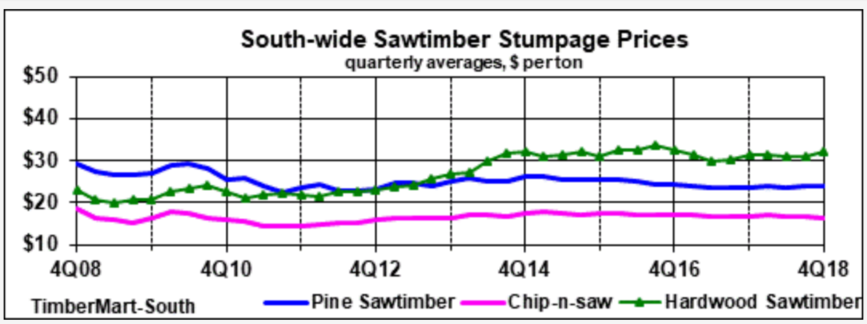

Figure 4.1: Graph and table for south-wide sawtimber stumpage prices over time in US$/ton. Anon. [2018]

4.1.2 Harvest Cost

To obtain the net return from harvesting a loblolly stand, the cost of harvest is subtracted off of the profit made. Profit for time period i is now known, as a value for p_i is set. More involved than the profit is the cost of harvest, which includes many separate costs as mentioned in chapter one. Note that the cost function accounts for some environmental requirements as money must be spent on efforts to make the forest a healthier, safer place. Accounting for the cost of harvest by finding all individual cost components may be incredibly time consuming and tedious. Yoder Lumber located in Millersburg, OH provides a strategy to approach this issue by hiring subcontractors to perform the actual harvesting duties. Yoder compensates these subcontractors using predetermined fixed rates based on the distance of the site from their mill. The further the site is, the more that Yoder pays. Subcontractors working on the closest sites are paid $115/mbf harvested while those working on the furthest sites are paid $125/mbf. The only other cost that Yoder must pay is the cost of transport for the harvested logs to reach their mill. This is known as the carrier rate. Yoder [2019] When considering the calculation of the cost of harvest for Yoder Lumber, a function for the net return from harvesting is simplified due to the involvement of just two fixed rates.

While Yoder Lumber typically works with hardwoods, using Yoder's exact values for determining the cost of harvest is not entirely accurate as loblolly pine is a softwood. A function for harvest costs can be more representative of the specific forest system and is just as simplified by using region-wide cost averages. The average cost of harvest, including the cost of loading and hauling, for the southern region of the United States in 2011 was $17.01/ton. Baker et al. [2013] This is equivalent to about $127.58/mbf, using the South Carolina Forestry Commission's conversion. The properties studied to create this average were predominantly pine stands. The average reported is a weighted average, accounting for a greater contribution of some cost components compared to others. The cost of hauling timber off-site was the largest component of overall harvest costs in the south, with labor costs as a close second. The reported average cost of harvest for 2011 is suspected to be lower than the average for all contractors due to the fact that the contractors who agreed to report on their harvesting costs "were selected based on their reputations for quality and consistent performance." Baker et al. [2013] These contractors are better with spending less on harvests as they know how to harvest more efficiently. However, the contractors selected are more likely to have accurate cost records, and thus provide a better understanding of harvesting costs in the south. Baker et al. [2013]

Adjusting the provided southern region average from 2011 to obtain a 2018 average may yield a usable cost value, but an investigation of logging costs over many years proves that the 2011 value from Baker et al. [2013] is believably too low. In order to look at the costs of logging in the eastern United States, from Michigan to Florida and then over to Texas, the Department of Forestry has tracked the costs of harvesting over a 20-year period for a variety of contractors that work primarily with pine stands. Reported average logging costs per ton between 1989 and 2009 ranged from \$18 to \$21 when adjusted to 2010 dollars, removing the effects of inflation. Figure 4.2 illustrates this trend, as follows. Stuart and Grace [2011] The solid line represents the adjusted costs. Thus, average costs ranged roughly from \$135/mbf to \$157.5/mbf. The average of these two values is \$146.25/mbf. Note that this value is larger than the cost provided by Baker et al. [2013].

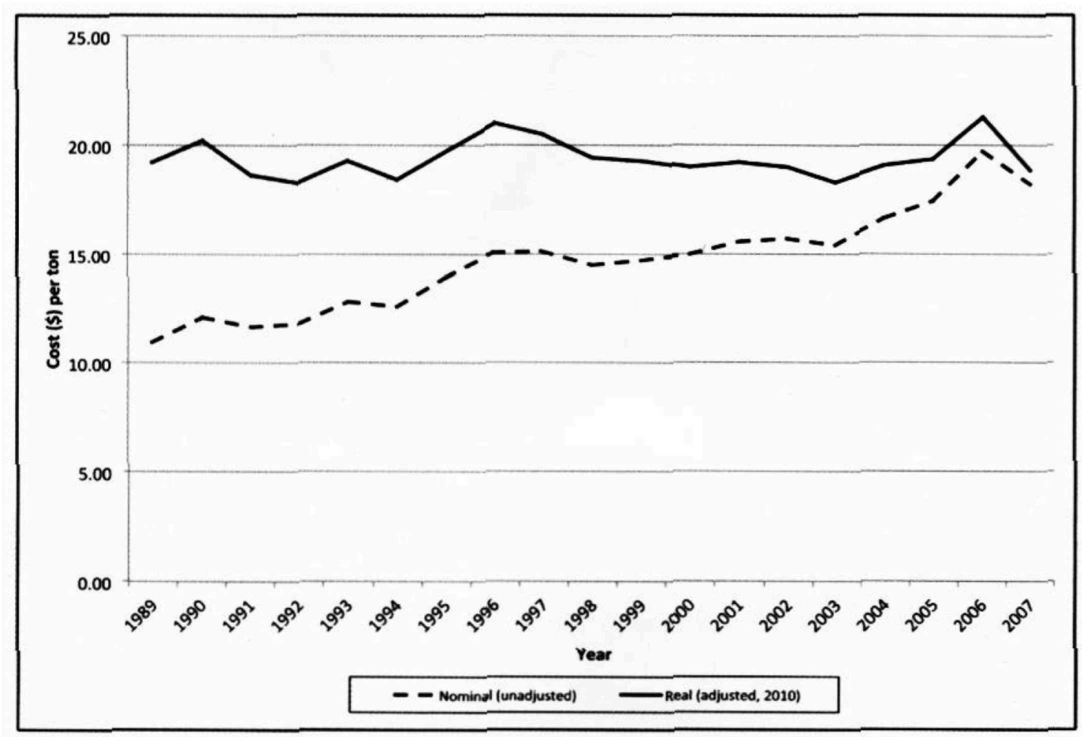

Figure 4.2: Graph of eastern harvesting costs over two decades in $/ton. Stuart and Grace [2011]

Adjusting $146.25/mbf so as to obtain a value for 2018 yields $167.33/mbf, and is an element of the cost of harvest used in the continuous model. Inflation rates for each year from 2011 to 2018 were found using Anon. [2019]. There is still one aspect of the harvest cost that will be incorporated into the model, addressing the dissemination of fixed costs as x_i increases.

Even though there are many components of the cost to harvest, there are often times quite a few fixed costs associated with harvesting a stand. As a stand increases in volume, these fixed costs are spread across a larger quantity of timber, driving the cost to harvest per mbf down. Jacobson [2008] Hence, if the volume in a stand is larger, it is cheaper per mbf to harvest the desired amount. This indicates that the harvesting cost per mbf decreases as the total stand volume increases. The inclusion of a multiplier, B, which scales the computed cost of $167.33/mbf, depending upon the present volume of timber, x_i, addresses this occurrence. This multiplier is structured as follows where M is a maximum mbf capacity in the stand. As the total volume of timber in the stand, x_i, increases, this multiplier gets closer to zero, causing a decrease in harvest cost. x_i is multiplied by 0.25 in B so as to avoid the overall cost value getting too close to zero. This produces a reduction in cost per mbf of 75% of 167.33 if x_i is near the maximum capacity. Note that if x_i is near zero, the cost will be close to the full 167.33. Without multiplying x_i by 0.25, when the stand gets close to full capacity, B gets super close to zero, causing the overall cost of harvest to be unrealistically close to zero.

$$B = \frac{M - 0.25x_i}{M} \text{ where } 0 \leq B \leq 1$$

$\boxed{\text{Set } c_i(x_i, u_i) = 167.33 * Bu_i}$

4.1.3 Net Return

The complete function for the net return from harvesting u_i from x_i in time period i, $A_i(x_i, u_i)$, is found below.

$\boxed{A_i(x_i, u_i) = p_i u_i - c_i(x_i, u_i) = 179.7u_i - 167.33 * Bu_i}$

4.1.4 Discount Factor

Considering the second part of the maximization equation for the continuous model, the variable α is a discount factor which adjusts the future value in a stand based off of today's market. In general, this variable accounts for inflation. The annual inflation rate in the United States in 2018 was 1.9%, which is also a rough average of annual inflation rates in the United States over the past ten years. Anon. [2019] To adjust the value of a stand one year in the future to this year's value considering an inflation rate of 1.9%, the α variable becomes $\frac{1}{1.019} = 0.98$. So, to know the current value of a stand, the value of the stand in one year is multiplied by 0.98. As all time periods are five years, $\alpha = 0.98^5 = 0.9$.

$\boxed{\text{Set } \alpha = 0.9}$

Each α will adjust the present value of the stand in the $i+1$ time period to the current price, adding value to the current stand. An updated maximization equation is provided below, displaying this idea through notation. The remainder of the model is also provided, with substitutions for any found variables and functions up to this point.

$$V_i(x_i) = \max_{u_i}[A_i(x_i, u_i) + \alpha V_{i+1}(x_{i+1})] \tag{4.1}$$

$$= \max_{u_i}[179.7u_i - 167.33 * Bu_i + 0.9V_{i+1}(x_{i+1})] \tag{4.2}$$

$$\text{where } x_{i+1} = x_i + g_i(x_i) - u_i \tag{4.3}$$

$$V_9(x_9) = F(x_9) \tag{4.4}$$

The pieces of the model left to establish are $g_i(x_i)$, the growth of the stand in timber volume in period i, $F(x_9)$, the value of the terminal timber volume, and M, the maximum mbf capacity of the stand. After M is obtained, we then have B.

4.1.5 Terminal Timber Value

$F(x_9)$ is needed during the last stage, $n = 8$, to find $V_8(x_8)$. By Equation (4.1), $V_9(x_9)$ must be known in order to find $V_8(x_8)$. Since $V_9(x_9) = F(x_9)$, a substitution can be made. To estimate the value of the terminal timber volume, or the present value of all available timber in the 9th time period, a price per mbf for all x_9 is determined. Then, $V_8(x_8)$ can be calculated using $F(x_9)$.

Assuming that the cost of harvest during the $n+1$ time period is \$167.33/mbf and is not effected by B, the net return from harvesting all available timber (x_9) is equivalent to $(179.7-167.33)x_9 = 12.37x_9$. Adjusting the constant 12.37 using an annual inflation rate of 1.9%, as all prices are still in 2018 dollars, yields $(1.019)^{45} * 12.37 = 28.854$. The inflation rate is compounded over 45 years because the time horizon is 40 years with five-year time periods, and at the $n+1$, or 9th stage, 45 years have passed.

$\boxed{\text{Set } F(x_9) = 28.854x_9}$

To calculate x_9 for $F(x_9)$, Equation (4.3) is utilized. Equation (4.3) implies $x_9 = x_8 + g_8(x_8) - u_8$. The growth function is needed for this calculation. The next section is an investigation of the growth of loblolly pine, needed to deduce a function for all $g_i(x_i)$, and to complete the continuous model.

4.1.6 Determining the Growth of Loblolly Pine

From one stage to the next, the total quantity of timber in a stand changes depending on how much is harvested and how much is grown (see Equation (4.3)). All variables that are needed to calculate the quantity of timber in the next stage - x_i, $g_i(x_i)$, and u_i - are measured in timber volume units (mbf). Thus, $g_i(x_i)$ is the quantity of thousand board feet that a loblolly stand of volume x_i gains in time period i through tree growth and death. Using data from loblolly stands in the southeastern United States, an annual percentage of the volume of loblolly that is gained through tree growth and death in a one-acre stand is found. This percentage is then used to find a growth percent over five years, and is multiplied by x_i to represent our final growth function.

In their 1963 publication, Nelson et al. [1963] discuss board foot growth in relation to variables which have major effect on the volume of a tree. These factors are stand age, stand density, and site index. All three terms are needed for a discussion of growth percentage, and are described below.

stand age

the age of the trees within a stand, assuming that the stand is even-aged.

stand density

to measure the density of a stand one can use what is called the basal area. Basal area is a measurement of the "amount of an area (usually an acre) occupied by tree stems." Anon. Another way of thinking about basal area would be to imagine if all trees in a stand were chopped off at a height of 4.5 feet. The roughly circular area within each tree at that height can then be calculated. Basal area is an estimation of the sum of all of these cross-sectional areas. A formula for basal area will be provided in this section as a function of the diameter of a tree at breast height (4.5 feet).

site index

a measurement of the productivity of a site. The site index is the average height in feet of the tallest trees in a stand, the dominant and codominant trees. Codominant trees make up the general canopy layer while the dominant trees are the tallest trees overall. The taller the dominant and codominant trees, the higher the site index. A large site index is an indication of good productivity. Yancey [2014]

By monitoring the growth of loblolly pine over ten years in three southeastern states, Nelson et al. [1963] provides a least-squares solution for annual board foot growth in terms of stand age (A), density (D), and site index (S). Note that solutions are in board feet, not mbf. The equation is provided below.

$$\begin{aligned}\text{Annual board-foot growth } &= 866.8 + 0.37917\left(\frac{10,000}{A}\right) - 23.7774(S) + 0.14207(S^2) \\ &- 17.08437(D) + 0.37445(SD) - 0.00071(SD^2) \end{aligned} \tag{4.5}$$

To build a growth function for the continuous model, $g_i(x_i)$, the annual board foot growth values provided for specific stand ages, basal areas per acre, and site indices are used to find a percentage of annual growth. By taking the annual board foot growth values in board feet given by Equation (4.5) (Figure 4.3) and dividing them by the total board feet in the stand, the percentage of annual growth is obtained. Thus, the total board feet in the stand needs to be found for all possible combinations of stand age, density, and site index. Lastly, the average of all growth percentages is computed and used to find a growth percentage over a five-year time period.

TABLE 1.—PERIODIC ANNUAL BOARD-FOOT GROWTH (INT. ¼-INCH RULE) PER ACRE OF TREES 9.6 INCHES D.B.H. AND LARGER TO A 7-INCH TOP DIAMETER INSIDE BARK AS RELATED TO AGE, TOTAL BASAL AREA PER ACRE, AND SITE INDEX

Age (years)	Total basal area per acre 30	40	50	60	70	80	90	100	110	120	130	140	150	160
	Board feet													
	Site 50													
30	177	168	153	130	100	63	19							
40	145	137	121	98	69	32								
50	126	118	102	79	50	13								
60	114	105	89	67	37	0								
70	104	96	80	58	28									
	Site 60													
30	201	225	241	248	246	236	217							
40	170	194	209	216	214	204	186							
50	151	175	190	197	195	185	167							
60	138	162	177	184	183	173	154							
70	129	153	168	175	174	164	145							
	Site 70													
30	254	310	357	394	420	437	444	441	427					
40	222	279	325	362	389	405	412	409	396					
50	203	260	306	343	370	386	393	390	377					
60	191	247	294	330	357	374	381	377	364					
70	182	238	285	321	348	365	372	368	355					
	Site 80													
30	335	424	502	568	623	666	699	719	729	727	714			
40	304	393	470	536	591	635	667	688	697	695	682			
50	285	374	451	518	572	616	648	669	678	676	663			
60	272	361	439	505	560	603	635	656	666	664	650			
70	263	352	430	496	551	594	626	647	657	655	641			
	Site 90													
30	445	566	675	771	854	924	982	1027	1059	1078	1084	1078		
40	413	535	643	739	822	893	950	995	1027	1046	1053	1046		
50	394	516	624	720	803	874	931	976	1008	1027	1034	1027		
60	382	503	612	708	791	861	919	963	995	1016	1021	1015		
70	373	494	603	699	782	852	910	954	986	1006	1012	1006		
	Site 100													
30	579	737	877	1002	1113	1211	1293	1362	1417	1457	1492	1493	1493	1476
40	547	705	845	971	1082	1179	1262	1331	1385	1425	1461	1461	1461	1445
50	528	686	826	952	1063	1160	1243	1312	1366	1406	1442	1442	1442	1426
60	516	674	813	939	1050	1147	1230	1299	1354	1394	1429	1430	1430	1413
70	507	665	804	930	1041	1138	1221	1290	1344	1385	1420	1421	1421	1404

Figure 4.3: Table of board foot growth values for all stand ages, basal areas per acre, and site indices in Nelson et al. [1963].

A couple timber harvesting management calculations are needed to first find the total board feet in a stand in terms of stand density and site index. One formula needed is for the calculation of basal area. The basal area of a standing tree in square feet, *BA*, is calculated using the diameter of the tree at breast height, in inches (*DBH*). Breast height is 4.5 feet from the base of a tree. Anon. The formula is provided below.

$$BA = 0.0055 * (DBH)^2 \tag{4.6}$$

Mentioned in chapter one, the International 1/4 inch rule calculates board feet (bf) using the following equation in terms of log length in feet, L, and the diameter inside the bark at the small end in inches, D. Cassens Equation (4.7) is the next timber management calculation needed.

$$\begin{aligned} bf = {} & 0.04976191LD^2 + 0.006220239L^2D - 0.1854762LD + 0.0002591767L^3 \\ & - 0.01159226L^2 + 0.04222222L \end{aligned} \tag{4.7}$$

This log rule will be used for calculating the total board feet in the stand.

Assuming that the diameter inside the bark of a loblolly tree at the small end is equivalent to the diameter of a loblolly tree at breast height, it is possible to make a substitution for D with DBH to calculate board feet given basal area and log length. Namely, if $DBH = D$, then DBH values can be used in a bf calculation as a substitute for D. Due to the fact that DBH is in terms of BA, the board foot calculation would then be in terms of BA and L instead of D and L. The substitution and simplification of the board foot formula is as follows.

Given $BA = 0.0055 * (DBH)^2 \implies DBH = \sqrt{\frac{BA}{0.0055}}$, and $DBH = D$,
we get the following:

$$\begin{aligned}
bf &= 0.04976191LD^2 + 0.006220239L^2D - 0.1854762LD + 0.0002591767L^3 \\
&\quad - 0.01159226L^2 + 0.04222222L \\
&= 0.04976191L(\sqrt{\frac{BA}{0.0055}})^2 + 0.006220239L^2(\sqrt{\frac{BA}{0.0055}}) \\
&\quad - 0.1854762L(\sqrt{\frac{BA}{0.0055}}) + 0.0002591767L^3 - 0.01159226L^2 + 0.04222222L \\
&= 9.04762L(BA) + 0.006220239L^2(\frac{BA}{0.0055})^{\frac{1}{2}} - 0.1854762L(\frac{BA}{0.0055})^{\frac{1}{2}} \\
&\quad + 0.0002591767L^3 - 0.01159226L^2 + 0.04222222L \\
&= 9.04762L(BA) + 0.0838736856L^2(BA)^{0.5} - 2.500960571L(BA)^{0.5} \\
&\quad + 0.0002591767L^3 - 0.01159226L^2 + 0.04222222L
\end{aligned}$$

Thus, the resulting equation is:

$$\begin{aligned}
bf &= 9.04762L(BA) + 0.0838736856L^2(BA)^{0.5} - 2.500960571L(BA)^{0.5} \\
&\quad + 0.0002591767L^3 - 0.01159226L^2 + 0.04222222L
\end{aligned} \tag{4.8}$$

Now that the total board feet in a stand can be found by knowing basal area and log length, one last change is made to substitute site index for log length. Log length is the length in feet of all merchantable sawtimber available from a standing tree, the non-merchantable sawtimber being the crown. To determine a relationship between site index and log length, live crown ratios are employed.

A live crown ratio is the ratio of crown height to total tree height. Tree vigor is said to increase as the live crown ratio increases. DeYoung For loblolly pine, the live crown ratio for a good tree vigor indication is 0.4. Zhao et al. [2009] Assuming that the trees in a stand are in good health, a live crown ratio of 0.4 implies that the percentage of the height of the trees in the stand that are merchantable sawtimber is 60%. Consequently, if SI is site index:

$$L = 0.6(SI) \tag{4.9}$$

By substitution of Equation (4.9) into Equation (4.8):

$$\begin{aligned}
bf &= 9.04762L(BA) + 0.0838736856L^2(BA)^{0.5} - 2.500960571L(BA)^{0.5} \\
&\quad + 0.0002591767L^3 - 0.01159226L^2 + 0.04222222L \\
&= 9.04762(0.6 * SI)(BA) + 0.0838736856(0.6 * SI)^2(BA)^{0.5} - 2.500960571(0.6 * SI)(BA)^{0.5} \\
&\quad + 0.0002591767(0.6 * SI)^3 - 0.01159226(0.6 * SI)^2 + 0.04222222(0.6 * SI) \\
&= 5.428572(SI)(BA) + 0.0301945268(SI)^2(BA)^{0.5} - 1.500576343(SI)(BA)^{0.5} \\
&\quad + 0.00005598(SI)^3 - 0.0041732136(SI)^2 + 0.02533333(SI)
\end{aligned}$$

A final equation for the total board feet in a loblolly stand in terms of stand density (basal area per acre) and site index is as follows.

$$\begin{aligned}
bf &= 5.428572(SI)(BA) + 0.0301945268(SI)^2(BA)^{0.5} - 1.500576343(SI)(BA)^{0.5} \\
&\quad + 0.00005598(SI)^3 - 0.0041732136(SI)^2 + 0.02533333(SI)
\end{aligned} \tag{4.10}$$

The total board foot volume can now be found for each BA and SI combination with the derived equation. The percentage of annual board foot growth is calculated by taking a given growth value in Figure 4.3 and dividing by total board feet for the correct site index and basal area. After computing all percentages, the average annual board foot growth percentage for southeastern loblolly pine is 1.74%. The board foot growth percentage becomes $(1.0174)^5 = 1.09 = 9\%$ after five years.

$\boxed{\text{Set } g_i(x_i) = 0.09x_i}$

Succeeding the calculation of all board foot growth percentages, the total board feet for each BA and SI combination can be compared to set an M value. Explained when discussing the B variable for the cost of harvest towards the beginning of section 4.1, M is the maximum mbf capacity of a stand. The maximum board foot value calculated using Equation (4.10) is 88795.17887 board feet which is 88.795 mbf. Rounding up, we set $M = 90$. Therefore, $B = \frac{M-0.25x_i}{M} = \frac{90-0.25x_i}{90}$.

$\boxed{\text{Set } B = \frac{90-0.25x_i}{90}}$

4.2 The Complete Model

Considering all found variables and functions for a one-acre loblolly pine stand, the complete continuous timber harvesting model is structured as follows. As the management time horizon is 40 years and each time period is five years, there are eight stages of the problem. Thus, $i = 1, 2, ..., 8$.

$$V_i(x_i) = \max_{u_i}[179.7u_i - 167.33(\frac{90 - 0.25x_i}{90})u_i + 0.9V_{i+1}(x_{i+1})] \tag{4.11}$$

$$\text{where } x_{i+1} = x_i + 0.09x_i - u_i \tag{4.12}$$

$$V_9(x_9) = F(x_9) = 28.854x_9 \tag{4.13}$$

4.3 The Dynamic Programming Process

Recall the process for solving a dynamic programming problem from the dynamic programming investigation in chapter three. In this section, Equation (4.11) is solved recursively. The optimal decisions for all states in the last stage, or time period, are examined first. As a reminder, each stage of the problem covers a five-year span, and the time horizon is 40 years long, beginning in 2018. Hence, the last time period, stage eight, covers years 2053 to 2058.

The present value of a stand in the last time period, $V_8(x_8)$, can be found by applying the assumption that $V_9(x_9) = F(x_9)$. The function $F(x_9)$ is substituted into the maximization equation for $V_9(x_9)$, as shown below.

$$
\begin{aligned}
V_8(x_8) &= \max_{u_8}[179.7u_8 - 167.33(\frac{90 - 0.25x_8}{90})u_8 + 0.9V_9(x_9)] \\
&= \max_{u_8}[179.7u_8 - 167.33(\frac{90 - 0.25x_8}{90})u_8 + 0.9 * 28.854x_9]
\end{aligned}
$$

Using Equation (4.12), we make a substitution for x_9.

$$
V_8(x_8) = \max_{u_8}[179.7u_8 - 167.33(\frac{90 - 0.25x_8}{90})u_8 + 0.9 * 28.854(x_8 + 0.09x_8 - u_8)]
$$

Simplifying the above yields:

$$
V_8(x_8) = \max_{u_8}[-13.5986u_8 + 0.4648x_8u_8 + 28.3058x_8]
$$

Optimal u_8 can now be found with a corresponding management decision for each feasible x_8. Just as in previous dynamic programming problems, all optimal

$V_8(x_8)$ are stored for each state. In this continuous problem, function optimization is utilized because every feasible u_8 cannot be tested discretely. The maximization of the simplified equation is performed by taking the derivative with respect to u_8 and setting it equal to zero, as follows.

$$\begin{aligned} &\frac{d}{du_8}[-13.5986u_8 + 0.4648x_8u_8 + 28.3058x_8] \\ &= -13.5986 + 0.4648x_8 = 0 \\ &\implies x_8 = 29.2569 \end{aligned}$$

While the derivative is a constant with respect to u_8, a maximum or minimum occurs at the endpoints $u_8 = 0$ and $u_8 = x_8$. The above also demonstrates that if $x_8 = 29.2569$, the derivative is equal to zero for all u_8. So, $V_8(x_8)$ is a constant function and all chosen u_8 are minimums and maximums. For $x_8 > 29.2569$, the derivative is a constant larger than zero. Hence, the maximum occurs at $u_8 = x_8$, where u_8 is the largest it can be. For any $x_8 < 29.2569$, the derivative is a constant less than zero. This means the maximum occurs at $u_8 = 0$, where u_8 is the smallest it can be.

Thus, if the total volume of timber in the stand is above 29.2569 mbf at the 8th stage, the best management decision to make would be to harvest everything, as harvesting more yields even more present value. If the total volume of timber in the stand at the 8th stage is below the critical point, then the harvesting level is not maximized, and it is more beneficial to leave the stand to be reassessed in the next stage. Therefore, for $0 \leq x_8 < 29.2569$ the stored optimal decision for every state is to leave the stand to grow, and for $x_8 \geq 29.2569$ the stored optimal decision for every state is to harvest everything. The two optimal decisions are for $u_8 = 0$ and $u_8 = x_8$.

1. For $0 \leq x_8 < 29.2569$, $u_8 = 0$ and $V_8(x_8) = 28.3058x_8$.

2. For $x_8 \geq 29.2569$, $u_8 = x_8$ and $V_8(x_8) = 14.7072x_8 + 0.4648(x_8)^2$.

By optimizing u_8, $V_8(x_8)$ is found and stored for each x_8. Each stored $V_8(x_8)$ is then used in the next step, moving backwards to the the 7th time period. In this stage, $V_7(x_7)$ is as follows.

$$V_7(x_7) = \max_{u_7}[179.7u_7 - 167.33(\frac{90 - 0.25x_7}{90})u_7 + 0.9V_8(x_8)]$$

For the first optimal decision in the last stage ($0 \leq x_8 < 29.2569$), harvesting no timber, a maximized function for $V_8(x_8)$ was found in terms of x_8. This is substituted into the $V_7(x_7)$ formula.

$$V_7(x_7) = \max_{u_7}[179.7u_7 - 167.33(\frac{90 - 0.25x_7}{90})u_7 + 0.9 * 28.3058x_8]$$

Again, using Equation (4.12), we get:

$$V_7(x_7) = \max_{u_7}[179.7u_7 - 167.33(\frac{90 - 0.25x_7}{90})u_7 + 0.9 * 28.3058(x_7 + 0.09x_7 - u_7)]$$

Through simplification and function optimization using the derivative with respect to u_7, the point at which to choose to harvest the stand or leave it alone to grow is found for this case. This point is $x_7 = 28.195$. Similar to the last stage, if $0 \leq x_7 < 28.195$ the stored optimal decision for every state is to leave the stand to grow, and for $x_7 \geq 28.195$ the stored optimal decision for every state is to harvest everything.

When following the second optimal decision in the last stage to the 7th stage

($x_8 \geq 29.2569$), the maximized quadratic function is substituted into the $V_7(x_7)$ formula. Substitutions using Equation (4.12) are also made.

$$V_7(x_7) = \max_{u_7}[179.7u_7 - 167.33(\frac{90 - 0.25x_7}{90})u_7$$
$$+ 0.9 * (14.7072(x_7 + 0.09x_7 - u_7) + 0.4648(x_7 + 0.09x_7 - u_7)^2)]$$

Note that the objective function, $V_7(x_7)$, is now quadratic, creating more difficulties in finding the management decisions for all x_7. Simplifying the above equation results in:

$$V_7(x_7) = \max_{u_7}[0.418325(u_7)^2 + (-0.8665 - 0.447x_7)u_7 + 14.4277x_7 + 0.497(x_7)^2]$$

Applying function optimization yields the following:

$$\frac{d}{du_7}[0.418325(u_7)^2 + (-0.8665 - 0.447x_7)u_7 + 14.4277x_7 + 0.497(x_7)^2]$$
$$= -0.8665 - 0.4471x_7 + 0.8367u_7 = 0$$
$$\implies u_7 = 1.0356 + 0.5344x_7$$

Thus, a critical point is found where $u_7 = 1.0356 + 0.5344x_7$. This u_7 value needs to be tested, along with the endpoints to find the overall maximum for the stage; the maximum may also occur at where $u_7 = 0$ or $u_7 = x_7$. After plugging all three u_7 values into $V_7(x_7)$ and graphing, $V_7(x_7)$ is found to be optimal for every state x_7 when $u_7 = 0$. (see $v(0, x)$, $v(1.0357 + 0.5344x, x)$, and $v(x, x)$ in Figure 4.4) The graph when $u_7 = 0$ is shown in red. That $v(0, x)$ has the largest coefficient on x^2 also indicates that u_7 is optimal for all states.

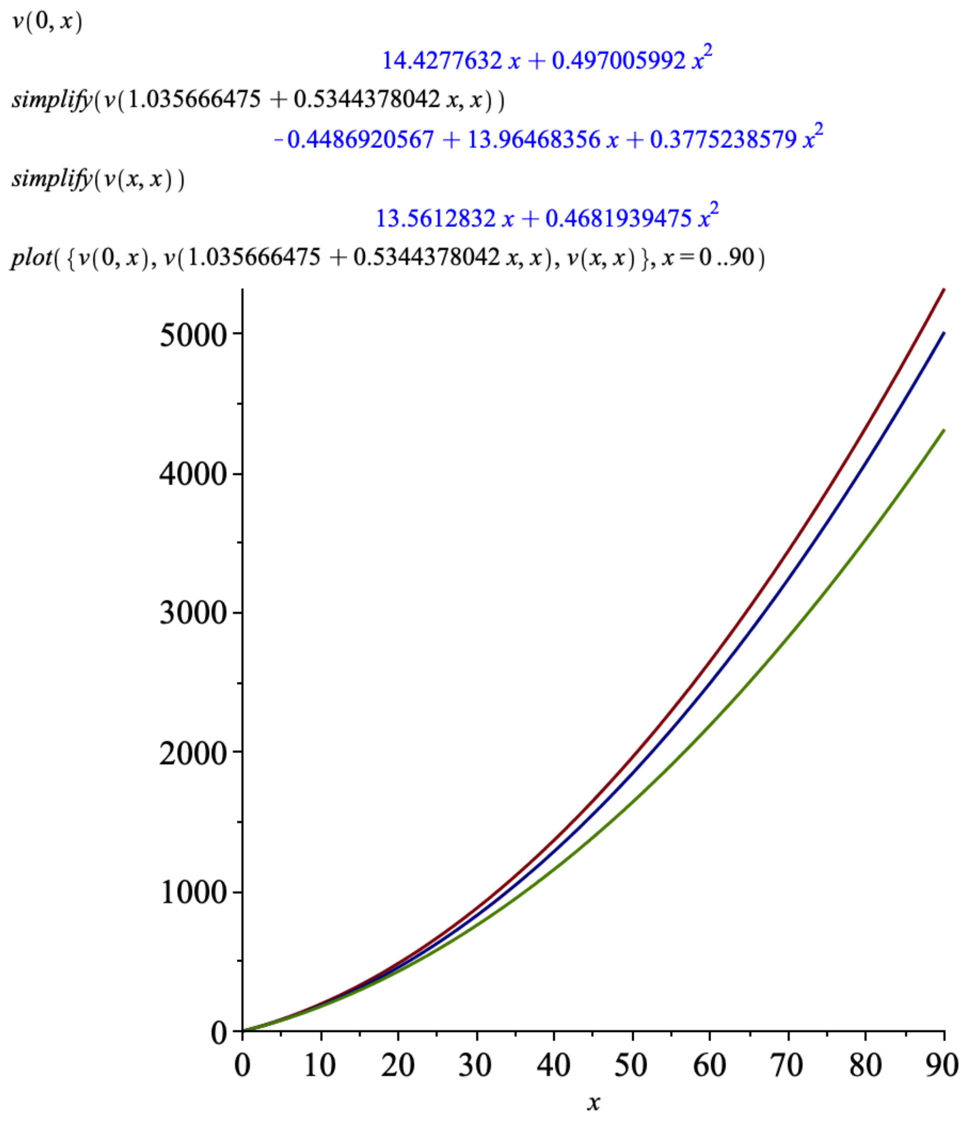

Figure 4.4: Equations and figure when following the second optimal decision in the last stage to the 7th stage.

Stepping back into the 6th stage follows the same optimization process, obtaining optimal management decisions for each state, with only two management decisions available. Each step backwards gets increasingly complex. This continuous model demonstrates the process of stepping through a dynamic programming problem through an application of a timber harvesting system with management decisions in each time period. By applying a dynamic programming model, the optimal present value over the entire time horizon can be found.

4.4 Further Works

While the structure and solving process of a dynamic programming problem are demonstrated, the application of the constructed models to real-world data is not. Future work may include the exploration and comparison of the constructed models and real-world data. The last model discussed, the continuous timber harvesting model, contains values and functions which attempt to replicate a specific stand in the southeastern United States. Hence, the accuracy of those values and functions can be tested.

One piece of the continuous model that would be interesting to expand upon would be the growth function. The growth rates of a species depends on many factors, some of those being tree age, basal area, site index, and more. An investigation of the plethora of factors impacting growth rates might further develop an understanding of timber growth rates and the influence of $g_i(x_i)$ on the continuous model. Starting with an extended look into the influencers of the growth rates of loblolly pine in the southeastern United States could lead to the improvement of solutions to the continuous model. It would be interesting to investigate and model other species in a timber production setting as well.

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